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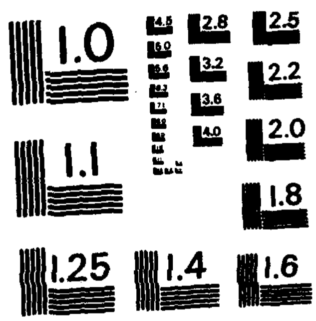

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A solution formulation of generalized Rayleigh-Ritz method is described and applied to two initial and boundary value problems of stress waves and structural dynamics in conjunction with finite element discretization. Excellent numerical results have been obtained for wave equations associated with lateral and longitudinal vibrations and with strong discontinuities.

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INTRODUCTION

This report describes a solution formulation for and its applications to initial boundary value problems of structural dynamics and stress waves. Excellent numerical results are stated in conjunction with finite element discretization. The basic concept of this approach is to establish a variational problem equivalent to a given initial boundary value problem, which is in general, non-self-adjoint, through the use of an adjoint field variable and the use of some large "spring" constants so that all the end conditions can be transformed into natural "boundary" conditions. Therefore, the shape functions used need not satisfy any end conditions a priori in solving the variational problem in the same manner as applying the Rayleigh-Ritz method for self-adjoint problems. This same concept was demonstrated in solving initial value problems in a paper delivered at the International Symposium on Numerical Methods in Engineering Science series in 1978 and later published in the Journal of Sound and Vibration.[1] In this present report, the formulation is extended to initial boundary value problems and the numerical results obtained are also encouraging.

In the section which follows immediately, two initial boundary value problems are stated. One is a longitudinal stress wave problem in a rod. There is a discontinuity in the initial data given. We wish to trace this discontinuity in the numerical solution using the present approach. The second problem is a beam vibration problem under a moving concentrated load.

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[1] J. J. Wu, "Solutions to Initial Value Problems by Use of Finite Elements - Unconstrained Variational Formulations," 1977 Journal of Sound and Vibration, 53, pp. 341-356.

This is a much more difficult problem since the partial differential equation is of fourth order and the force is singular in nature. In the next section, variational problems equivalent to the given initial boundary problems are established. The finite element discretization procedures are then briefly recaptured. Lastly, numerical results are presented with some comments.

INITIAL BOUNDARY VALUE PROBLEMS

Two initial boundary problems of structural dynamics will be stated in this Section. The first one is of longitudinal elastic stress wave in a rod with a sudden change in initial conditions. The second one is concerned with lateral vibrations of a Euler-Bernoulli beam subjected to a moving concentrated load.

Longitudinal Stress Wave in a Rod

The rod is fixed at one end and free at the other end. The discontinuity data arises from the initial linear displacement, corresponding to a constant stress, due to a force applied at the "free" end. This force suddenly disappears at time zero causing a stress discontinuity at the free end. The differential equation can be written as:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \quad ; \quad \begin{matrix} 0 < x < \ell \\ 0 < t < T \end{matrix} \tag{1}$$

with

$$a^2 = E/\rho \tag{2}$$

where $u = u(x,t)$ is the axial displacement; x,t are the coordinates in axial direction and in time, respectively; ρ, E are density and Young's modulus, respectively, of the rod material; ℓ denotes length of the rod; and T denotes some finite time of interest.

For the boundary conditions, we have

$$u(0,t) = 0$$

and

$$\frac{\partial u}{\partial x}(1,t) = 0 \tag{3}$$

The dynamics of the problem are due to the initial conditions. It is assumed that the rod is stretched to a linear displacement by a force P which vanishes at time $t > 0$ (see Figure 1). The initial velocity of the rod is assumed to be zero. Thus

$$u(x,0) = \frac{P}{AE}x \quad ; \quad \text{and}$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \tag{4}$$

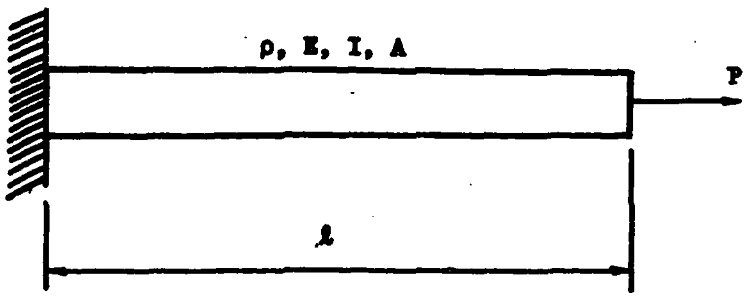

Figure 1. A Rod Fixed at One End and Subjected to a Load P, which is Suddenly Released at Time Zero.

It is convenient to use dimensionless parameters. Let

$$u^* = u/\ell \quad , \quad x^* = x/\ell \quad , \quad t^* = t/T \tag{5}$$

Then, Eq. (1) in dimensionless form is

$$\frac{\partial^2 u^*}{\partial x^{*2}} = b^2 \frac{\partial^2 u^*}{\partial t^{*2}} \quad , \quad \begin{array}{l} 0 < x^* < 1 \\ 0 < t^* < 1 \end{array} \tag{6}$$

where

$$b^2 = \frac{1}{a^2} \left(\frac{\ell}{T}\right) \tag{7}$$

The boundary conditions become

$$u^*(0,t^*) = 0 \quad , \quad \frac{\partial u^*}{\partial x^*}(1,t^*) = 0 \tag{8}$$

and

$$u^*(x,0) = P^*x^* \quad ; \quad \frac{\partial u^*}{\partial t^*}(x^*,0) = 0 \tag{9}$$

where

$$P^* = \frac{P}{AE} \tag{10}$$

is the force in dimensionless form.

The stated problem in dimensionless form combines Eqs. (6), (8), and (9) with the new dimensionless parameters related to physical counterparts by Eqs. (5), (6), and (10). To simplify writing, we shall drop the asterisks (*) in Eqs. (6), (8), and (9), and rewrite them as

$$u'' - b^2\ddot{u} = 0 \quad ; \quad \begin{matrix} 0 < x < 1 \\ 0 < t < 1 \end{matrix} \tag{6'}$$

$$u(0,t) = 0 \quad ; \quad u'(1,t) = 0 \tag{8'}$$

$$u(x,t) = Px \quad ; \quad \dot{u}(x,0) = 0 \tag{9'}$$

where a prime (') indicates differentiation with respect to x and a dot (·), with respect to t.

Beam Vibrations Under Moving Loads

Let us consider the differential equation of a uniform Euler-Bernoulli beam subjected to a moving, concentrated force.

$$EIy'''' + \rho A\ddot{y} = \bar{P}\delta(x_p - x) \quad \begin{matrix} 0 < x < \ell \\ 0 < t < T \end{matrix} \tag{11}$$

where

E, ρ = Young's modulus, density of the beam material

I, A = second moment, area of the beam's cross-section

ℓ = length of the beam

$y = y(x,t)$ = beam deflection

x, t = coordinates in beam's axial direction and in time

P = magnitude of the concentrated force

$\delta(x)$ = Dirac delta function

$x_p = x_p(t)$ = location of P

T = some finite time of interest

Again it will be convenient to employ nondimensional parameters and equations. These will be introduced by way of Eq. (11). Thus, let

$$y^* = y/\ell \quad , \quad x^* = x/\ell \quad , \quad t^* = t/T \tag{12}$$

Using Eq. (12) in Eq. (11), one has

$$y^{*''''} + \gamma^2 y^* = Q\bar{\delta}(x_p{}^* - x^*) \qquad \begin{matrix} 0 < x^* < 1 \\ 0 < x^* < 1 \end{matrix} \tag{13}$$

where

$$\gamma = \frac{c}{T} \quad , \quad c^2 = \frac{\rho A \ell^4}{EI} \quad , \quad Q = P^* = \frac{P\ell^2}{EI} \tag{14}$$

Also note in Eq. (13) that the differentiations are now with respect to the nondimensionalized variables x^* ant t^*. From now on, we shall use Eq. (13) with the asterisks dropped altogether.

$$y'''' + ky + \gamma^2 \ddot{y} = Q\bar{\delta}(x_p - x) \qquad \begin{matrix} 0 < x < 1 \\ 0 < t < 1 \end{matrix} \tag{15}$$

VARIATIONAL PROBLEMS - A GENERALIZED RAYLEIGH-RITZ METHOD

For the stress wave problem in the previous section, consider a variational problem.

$$\delta I_o = 0 \tag{16a}$$

with

$$I_o = I_o(u,v) = \int_0^1 \int_0^1 (-u'v'+b^2\dot{u}\dot{v})dxdt \tag{16b}$$

where u(x,t) and v(x,t) are said to be adjoint to each other. It is a simple matter to see that this problem is an indeterminate one. However, the functional of Eq. (16b) can be modified to a variational problem which is equivalent to the boundary/initial problem of Eqs. (6'), (8'), and (9'). Thus consider

$$\delta I = 0 \tag{17a}$$

with

$$\begin{aligned} I = I(u,v) = \int_0^1 \int_0^1 (-u'v'+b^2\dot{u}\dot{v})dxdt \\ + k_1\int_0^1 u(0,t)v(0,t)dt \\ + k_2b^2\int_0^1 [u(x,0) - u_o(x)]v(x,1)dx + b^2 \int_0^1 u_1(x)v(x,0)dx \end{aligned} \tag{17b}$$

We shall take the first variation of the function I(u,v) of Eq. (17b) in such a manner that δv is completely arbitrary while δu is set to zero identically. Hence, by means of integration-by-parts, one has

$$(\delta I)_{\delta u=0} = \int_0^1 \int_0^1 (u'' - b^2\ddot{u})\delta v dx dt$$

$$- \int_0^1 u'(1,t)\delta v(1,t)dt$$

$$+ \int_0^1 [u(0,t) + k_1 u(0,t)]\delta v(0,t)dt$$

$$+ b^2 \int_0^1 \{\dot{u}(x,1) + k_2[u(x,0) - u_0(x)]\}\delta v(x,1)dx$$

$$- b^2 \int_0^1 [\dot{u}(x,0) - u_1(x)]\delta v(x,0)dx = 0 \quad (18)$$

The fact that $\delta v(x,t)$ is completely arbitrary enables us to conclude from Eq. (18) that

$$u'' - b^2 u = 0 \quad ; \quad \begin{matrix} 0 < x < 1 \\ 0 < t < 1 \end{matrix} \quad (19a)$$

and

$$\begin{aligned} u'(1,t) &= 0 \\ u'(0,t) + k_1 u(0,t) &= 0 \\ u(x,1) + k_2[u(x,0) - u_0(x)] &= 0 \\ u(x,0) - u_1(x) &= 0 \end{aligned} \quad (19b)$$

It is then observed that the initial boundary value problem defined by Eqs. (19a) and (19b) reduces to that of Eqs. (6'), (8'), and (9') if one lets k_1 and k_2 go to infinity* (and with $u_0(x) = Px$ and $u_1(x) = 0$). This fact suggests that the variational problem of Eqs. (17a) and (17b) can be used as a basis of a finite element discretization for the approximate solutions to the original initial boundary problem. It should be noted that all the auxiliary

*This process is sometimes referred to as the penalty function method. See, for example, Reference 2.

[2]D. G. Luenberger, Optimization by Vector Space Method, John Wiley, 1969, p. 302.

conditions in Eqs. (19a) and (19b) are the so-called natural boundary conditions. They are the consequence of the variational problem - just like the differential equation itself. For this reason, the above solution is referred to as a Generalized Rayleigh-Ritz Method.

By a similar process, one can establish a variational problem for the vibration problem of a beam under a moving load. In this case, one has

$$\delta I = \int_0^1 \int_0^1 [u''\delta v'' - \dot{u}\delta\dot{v} - \bar{\delta}(x-\bar{x})\delta v]dxdt$$
$$+ \int_0^1 [k_1 u(0,t)\delta v(0,t) + k_2 u'(0,t)\delta v'(0,t)$$
$$+ k_3 u(1,t)\delta v(1,t) + k_4 u'(1,t)\delta v'(1,t)]dt$$
$$+ \int_0^1 [k_5 u(x,0)\delta v(x,1) + k_6 \dot{u}(x,0)\delta v(x,0)]dx = 0 \tag{20}$$

Through integrations-by-parts,

$$\delta I = \int_0^1 \int_0^1 [u'''' + \ddot{u} - \bar{\delta}(x-\bar{x})]\delta v(x,t)dxdt$$
$$+ \int_0^1 \{[k_1 u(0,t) + u'''(0,t)]\delta v(0,t) + [k_2 u'(0,t) - u''(0,t)]\delta v'(0,t)$$
$$+ [k_3 u(1,t) - u'''(1,t)]\delta v(1,t) + [k_4 u'(1,t) + u''(1,t)]\delta v'(1,t)\}dt$$
$$+ \int_0^1 \{[k_5(u(x,0)-0) - \dot{u}(x,t)]\delta v(x,1) + (k_6+1)[\dot{u}(x,0)-0]\delta v(x,0)\}dx = 0 \tag{21}$$

The original differential equation and the boundary and initial conditions are recovered from the equation above due to the arbitrariness of the variations $\delta(x,t)$ and by properly selecting the values of k_i,s, $i = 1,2,\ldots,6$.

FINITE ELEMENT DISCRETIZATION

Only essential features will be stated in the finite element discretizations here. The region of a unit square ($0 < x < 1$; $0 < t < 1$) is further divided into KxL rectangles by taking K divisions in x direction and L divisions in t direction. Local coordinates (ξ,η) in each (i,j)th element are related to (x,t) by these equations:

$$\begin{aligned} \xi &= \xi^{(i)} = Kx - i + 1 \\ \eta &= \eta^{(j)} = Lt - j + 1 \end{aligned} \tag{22}$$

Within each element, the unknown function u(x,t) is replaced by the approximation:

$$\begin{aligned} u_{(i,j)}(\xi,\eta) &= \underset{\sim}{a}^T(\xi,\eta)\; U_{(i,j)} \\ \delta v_{(i,j)}(\xi,\eta) &= \underset{\sim}{a}^T(\xi,\eta)\; \delta V_{(i,j)} \end{aligned} \tag{23}$$

where $a(\xi,\eta)$ is the shape function vector and $U_{(i,j)}$, $\delta V_{(i,j)}$ are the generalized coordinates. The specific form of $a(\xi,\eta)$ employed here is such that each one of the sixteen components is:

$$a_k(\xi,\eta) = b_i(\xi)b_j(\eta) \quad , \quad \begin{aligned} k &= 1,2,\ldots\ldots,16 \\ i,j &= 1,2,3,4 \end{aligned} \tag{24}$$

with

$$\begin{aligned} b_1(\epsilon) &= 1 - 3\xi^2 + 2\xi^3 \\ b_2(\xi) &= \xi - 2\xi^2 + \xi^3 \\ b_3(\xi) &= 3\xi^2 - 2\xi^3 \\ b_4(\xi) &= -\xi^2 + \xi^3 \end{aligned} \tag{25}$$

and the relations between index k and the pair (i,j) are given in Table I.

TABLE I. RELATIONSHIP BETWEEN (i,j) AND k IN EQUATION (24)

k	(i,j)	k	(i,j)
1	(1,1)	9	(1,3)
2	(2,1)	10	(2,3)
3	(1,2)	11	(1,4)
4	(2,2)	12	(2,4)
5	(3,1)	13	(3,3)
6	(4,1)	14	(4,3)
7	(3,2)	15	(3,4)
8	(4,2)	16	(4,4)

Using Eqs. (22) through (25) in Eq. (17) and the fact that $V_{(i,j)}$ is completely arbitrary, the matrix equations for the unknowns $U_{(i,j)}$ can be routinely assembled and solved. Further details will be omitted here.

NUMERICAL RESULTS AND DISCUSSION

Some of the numerical results are presented in this section. For the stress wave problem*, Table II provides solutions of $v(x,t)$, $\partial u/\partial x(x,t)$ and $\partial u/\partial t(x,t)$ for $x = 0, 0.1, 0.2, \ldots 1.0$ and for $t = 0, 0.5, 1.0, 1.5,$ and 2.0. During this time interval, the original displacement has gone through a complete sign reversal as shown in Figure 2. This particular set of data was obtained by taking $K = 10$ and $L = 1$ with restart procedures, i.e., the final

*For exact solution to this problem, see for example, Reference 3.

[3] L. S. Jacobson and R. S. Ayre, Engineering Vibrations, McGraw-Hill, 1965, pp. 472-474.

TABLE II. SOLUTIONS TO THE STRESS WAVE PROBLEM OF EQS. (6'), (8') and (9') WITH b = 1.0, P = 1.0.
(PART 1)

	Data at Time t = 0.0			Data at Time t = 0.50			Data at Time t = 1.00		
x	u(x,t)	∂u/∂x	∂u/∂t	u(x,t)	∂u/∂x	∂u/∂t	u(x,t)	∂u/∂x	∂u/∂t
0.0	0.00000 (0.00000)	1.00000 (1.00000)	0.00000 (0.00000)	−0.00000 (0.00000)	0.99861 (1.00000)	0.00000 (0.00000)	0.00000 (0.00000)	−0.11718 (0.00000)	0.00000 (0.00000)
0.10	0.10000 (0.10000)	1.00000 (1.00000)	0.00000 (0.00000)	0.09998 (0.10000)	0.99740 (1.00000)	−0.00142 (0.00000)	−0.01236 (0.00000)	0.18416 (0.00000)	−1.07661 (−1.00000)
0.20	0.20000 (0.20000)	1.00000 (1.00000)	0.00000 (0.00000)	0.19994 (0.20000)	0.99113 (1.00000)	−0.00618 (0.00000)	0.00259 (0.00000)	−0.10702 (0.00000)	−1.09024 (−1.00000)
0.30	0.30000 (0.30000)	1.00000 (1.00000)	0.00000 (0.00000)	0.29949 (0.30000)	0.97359 (1.00000)	−0.01886 (0.00000)	−0.00026 (0.00000)	−0.00077 (0.00000)	−0.92271 (−1.00000)
0.40	0.40000 (0.40000)	1.00000 (1.00000)	0.00000 (0.00000)	0.40354 (0.40000)	1.06038 (1.00000)	0.07965 (0.00000)	0.00035 (0.00000)	−0.00953 (0.00000)	−1.03476 (−1.00000)
0.50	0.50000 (0.50000)	1.00000 (1.00000)	0.00000 (0.00000)	0.49976 (0.50000)	0.47813 (1.00000)	−0.54638 (0.00000)	−0.00036 (0.00000)	−0.00287 (0.00000)	−0.95721 (−1.00000)
0.60	0.60000 (0.60000)	1.00000 (1.00000)	0.00000 (0.00000)	0.49785 (0.50000)	0.02932 (0.00000)	−0.96310 (−1.00000)	0.00042 (0.00000)	−0.00334 (0.00000)	−1.04064 (−1.00000)
0.70	0.70000 (0.70000)	1.00000 (1.00000)	0.00000 (0.00000)	0.50081 (0.50000)	−0.02913 (0.00000)	−1.06490 (−1.00000)	−0.00039 (0.00000)	0.00046 (0.00000)	−0.95841 (−1.00000)
0.80	0.80000 (0.80000)	1.00000 (1.00000)	0.00000 (0.00000)	0.49983 (0.50000)	0.00258 (0.00000)	−0.95849 (−1.00000)	0.00041 (0.00000)	−0.00086 (0.00000)	−1.04185 (−1.00000)
0.90	0.90000 (0.90000)	1.00000 (1.00000)	0.00000 (0.00000)	0.50019 (0.50000)	0.00036 (0.00000)	−1.03927 (−1.00000)	−0.00040 (0.00000)	−0.00012 (0.00000)	−0.95859 (−1.00000)
1.00	1.00000 (1.00000)	0.00000 (0.00000)	0.00000 (0.00000)	0.49982 (0.50000)	−0.00000 (0.00000)	−0.96235 (−1.00000)	0.00041 (0.00000)	0.00000 (0.00000)	−1.04164 (−1.00000)

*Figures in parentheses indicate exact solutions.

TABLE II. SOLUTIONS TO THE STRESS WAVE PROBLEM OF EQS. (6'), (8'), and (9') with b = 1.0, P = 1.0.
(PART 2)

	Data at Time t = 1.50			Data at Time t = 2.00		
x	u(x,t)	∂u/∂x	∂u/∂t	u(x,t)	∂u/∂x	∂u/∂t
0.0	0.00000 (0.00000)	−0.99528 (−1.00000)	0.00000 (0.00000)	0.00000 (0.00000)	−0.02019 (−1.00000)	−0.00000 (0.00000)
0.10	−0.9975 (−0.10000)	−1.01418 (−1.00000)	0.00235 (0.00000)	−0.10003 (−0.10000)	−0.98040 (−1.00000)	0.00165 (0.00000)
0.20	−0.19988 (−0.20000)	−0.93268 (−1.00000)	−0.04972 (0.00000)	−0.19992 (−0.20000)	−0.01911 (−1.00000)	−0.00119 (0.00000)
0.30	−0.30222 (−0.30000)	−1.13452 (−1.00000)	0.09812 (0.00000)	−0.30008 (−0.30000)	−0.97758 (−1.00000)	0.00610 (0.00000)
0.40	−0.39324 (−0.40000)	−0.62190 (−1.00000)	−0.02522 (0.00000)	−0.39981 (−0.40000)	−0.00984 (−1.00000)	−0.00308 (0.00000)
0.50	−0.49607 (−0.50000)	0.19515 (0.00000)	−0.40290 (0.00000)	−0.50035 (−0.50000)	−0.98060 (−1.00000)	−0.02259 (0.00000)
0.60	−0.51154 (−0.50000)	0.00219 (0.00000)	−1.14247 (−1.00000)	−0.59945 (−0.60000)	−1.01522 (−1.00000)	0.02001 (0.00000)
0.70	−0.49849 (−0.50000)	−0.00219 (0.00000)	−1.05659 (−1.00000)	−0.69929 (−0.70000)	0.84502 (−1.00000)	−0.05514 (0.00000)
0.80	−0.50034 (−0.50000)	−0.02908 (0.00000)	−0.92639 (−1.00000)	−0.80180 (−0.80000)	−0.84502 (−1.00000)	0.07546 (0.00000)
0.90	−0.49935 (−0.50000)	−0.01126 (0.00000)	−1.03406 (−1.00000)	−0.90529 (−0.90000)	−1.19080 (−1.00000)	−0.23643 (0.00000)
1.00	−0.50051 (−0.50000)	−0.00000 (0.00000)	−0.93441 (−1.00000)	−0.98802 (−1.0000)	−0.00000 (−1.00000)	0.31500 (0.00000)

*Figures in parentheses indicate exact solutions.

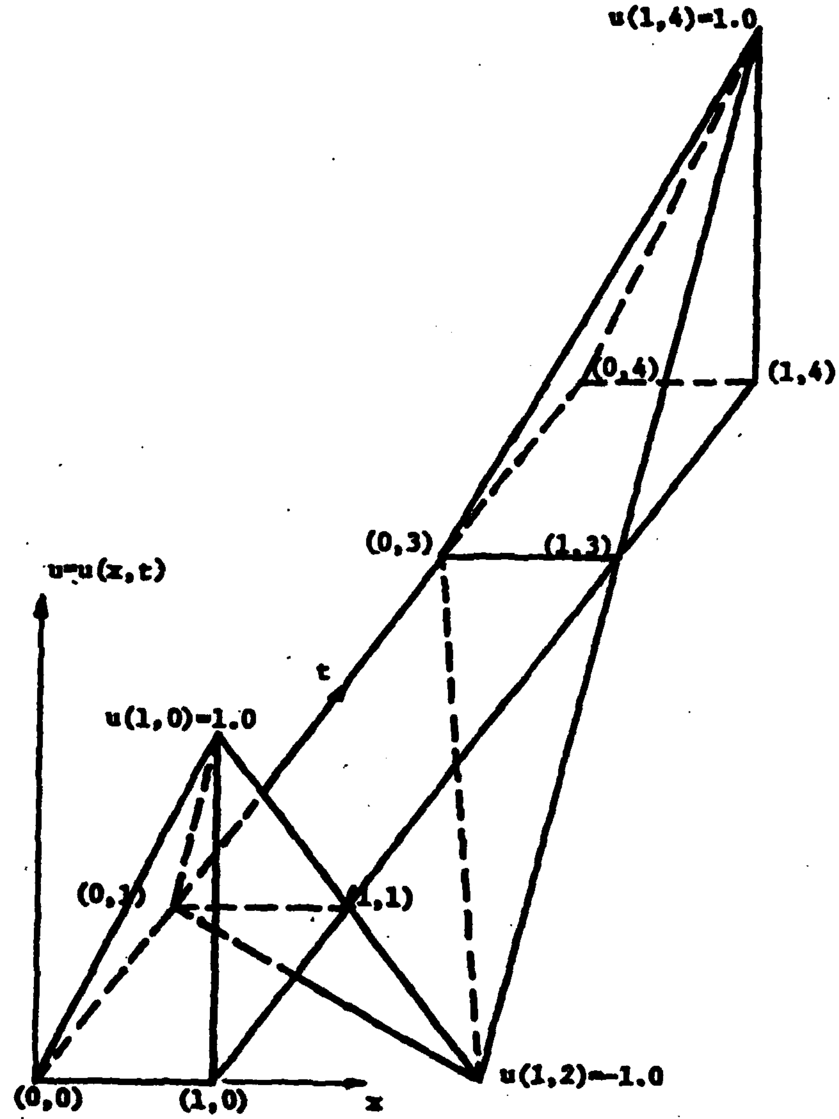

Figure 2. Exact Solution Surface u=u(x,t) for the Stress Wave Problem of Eqs. (6'), (8'), and (9') in the Region: $0 < x < 1$ and $0 < t < 4$ (with $P = 1.0$ and $b = 1.0$).

solution in the first time step was taken as the initial condition of the next step in time, and so on. Values of the exact solutions are given in parentheses. Excellent agreement is observed. The fact that the discontinuity of the solution follows along without much oscillation is worth mentioning.

For the beam vibration problem with a moving force, some typical numerical solutions are given in Tables III and IV. The moving concentrated force is assumed to travel at a constant velocity c (although this is not at all a restriction for the present method) such that

$$x(t) = ct$$

where c is dimensionless velocity. For small c, c = 0.0001, and the displacement solutions become those of static deflections as shown in Table III. For a large c (compared with unity), c = 10, and solutions show dynamic effects as indicated in Table IV. As a comparison, solutions obtained by the Fourier series and Laplace transform method[4] are given in parentheses. Good agreement exists even in cases with considerable dynamic effect.

In conclusion, this report has demonstrated through examples of structural dynamics an approximate solution formulation (which is both a weighted method and a variational problem), the finite element implementation, and some favorable numerical results. Although only linear problems have been mentioned, an application to solutions of non-linear problems is now being investigated.

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[4]L. Fryba, Vibrations of Solids and Structures Under Moving Load, Noordhoff, 1971.

TABLE III. SOLUTIONS $u(x,t)$ TO THE MOVING FORCE PROBLEM OF EQ. (15) WITH $Q = 1.0$ AND FIXED END CONDITIONS AT $x = 0$. (For very low velocity, $\gamma = 10^{-4}$ in Eq. (14))

x	t = 0.0	t = 0.20	t = 0.40	t = 0.60	t = 0.80	t = 1.00
0.0	0.000000 (Given)	0.000000 (0.000000)	0.000000 (0.000000)	0.000000 (0.000000)	0.000000 (0.000000)	−0.000000 (0.000000)
0.20	0.000001 (Given)	0.008533 (0.009534)	0.011999 (0.012000)	0.010665 (0.010667)	0.006125 (0.006133)	−0.000097 (0.000000)
0.40	−0.000001 (Given)	0.012001 (0.012000)	0.019206 (0.019199)	0.018157 (0.018134)	0.010763 (0.010666)	0.001174 (0.000000)
0.60	−0.000000 (Given)	0.010668 (0.010667)	0.018137 (0.018133)	0.019214 (0.019201)	0.012057 (0.012000)	0.000691 (0.000000)
0.80	0.000001 (Given)	0.006133 (0.006134)	0.010664 (0.010666)	0.011990 (0.012000)	0.008491 (0.008533)	−0.000509 (0.000000)
1.00	−0.000000 (Given)	0.000000 (0.000000)	0.000000 (0.000000)	0.000000 (0.000000)	0.000000 (0.000000)	0.000000 (0.000000)

Solutions in parentheses based on formulas from Reference 4.

TABLE IV. SOLUTIONS $u(x,t)$ TO THE MOVING FORCE PROBLEM OF EQ. (15) WITH $Q = 1.0$ AND FIXED END CONDITIONS AT $x = 0$.
(For very low velocity, $\gamma = 10$ in Eq. (14))

x	t = 0.0	t = 0.20	t = 0.40	t = 0.60	t = 0.80	t = 1.00
0.0	−0.000000 (Given)	−0.000000 (0.000000)	−0.000000 (0.000000)	−0.000000 (0.000000)	−0.000000 (0.000000)	−0.000000 (0.000000)
0.20	−0.000001 (Given)	0.000482 (0.000467)	0.001387 (0.001345)	0.002151 (0.002046)	0.002944 (0.002534)	0.007191 (0.003643)
0.40	−0.000001 (Given)	−0.000077 (−0.000082)	0.001109 (0.001109)	0.002717 (0.002704)	0.004364 (0.004375)	0.004877 (0.004463)
0.60	−0.000000 (Given)	0.000001 (0.000025)	−0.000320 (−0.000311)	0.001493 (0.001504)	0.003110 (0.003177)	0.005067 (0.005601)
0.80	−0.000001 (Given)	0.000003 (−0.000012)	−0.000013 (0.000002)	−0.000964 (−0.000942)	0.001308 (0.001257)	0.005751 (0.005464)
1.00	−0.000000 (Given)	−0.000000 (0.000000)	0.000000 (0.000000)	−0.000000 −(0.000000)	−0.000000 (0.000000)	−0.000000 (0.000000)

Solutions in parentheses based on formulas from Reference 4.

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2. D. G. Luenberger, Optimization by Vector Space Method, John Wiley, 1969, p. 302.

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